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PHASE AND GROUP VELOCITIES OF WATER WAVES - DESCRIPTION OF WAVE DEFORMATION USING FRACTIONAL DERIVATIVES

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Since 1980's it has been known that there exists a speed higher than that of light in a vacuum ($2.9979 \times 10^8\ m/s$). It is also known that a duck swims at about 0.7 m per second whose distance is only twice as large as the body, while a whirligig beetle, which is a small insect living on a water surface, can swim longer than several ten times of its size in a second. What is the difference between the speeds of two kinds of creatures? To know the above two interesting matters correctly, it is necessary to get a deep understanding of traveling waves of an arbitrary shape.

Some specific behaviors of wave propagation depend on the dispersion relation between the frequency and wavenumber of the wave. Most of the discussions on wave propagation have almost been made on monochromatic waves, i.e., on those of one wavenumber or one frequency. Waves of an arbitrary shape are expressed by the superposition over many waves of different wavenumbers.

We consider a superposition of many water waves with various k. The surface elevation $\eta(x,t)$ is written as

$$\eta(x,t) = \int A(k)\mathrm{e}^{i(kx-\omega t)}\mathrm{d}k \qquad (1)$$

where $\omega/k\ (= c)$ is called the *phase velocity*. The frequency ω depends on the wavenumber k. For gravity and capillary waves the dependence is in the dimensionless form as $\omega^2 = k\tanh k$ for gravity waves and $\omega^3 = k^3\tanh k$ for capillary waves, respectively. It should be noted that $\tanh kh \approx 1$ for deep water waves. On the other hand, another kind of velocity called the *group velocity* c_g defined as $\mathrm{d}\omega = \mathrm{d}k$ plays an important part in the investigation of the energy transport due to wave motion. It is important to notice the difference between the two. We first derive the equation governing the gravity wave. From eq.(1) the equation describing the wave development is written as

$$\frac{\partial \eta}{\partial t} + \frac{1}{\sqrt{2}}\left(I^{-\frac{1}{2}} - K^{-\frac{1}{2}}\right)\eta = 0 \qquad (2)$$

Two operators $I^{-\frac{1}{2}}$ and $K^{-\frac{1}{2}}$ are called the fractional derivatives of order $1/2$. The operators are written in general as

$$I^{-\lambda}f(x) = \int_{-\infty}^{\infty} \hat{f}(k)|k|^{\lambda}\, \mathrm{e}^{\frac{\pi}{2}\mathrm{i}\lambda \mathrm{sgn}k}\mathrm{e}^{\mathrm{i}kx}\mathrm{d}k$$

$$K^{-\lambda}f(x) = \int_{-\infty}^{\infty} \hat{f}(k)|k|^{\lambda}\, \mathrm{e}^{-\frac{\pi}{2}\mathrm{i}\lambda \mathrm{sgn}k}\mathrm{e}^{\mathrm{i}kx}\mathrm{d}k$$

where $I^{-\lambda}$ and $K^{-\lambda}$ are the Riemann-Liouville and Weyl derivatives of order λ, respectively, and sgn stands for the sign function.

We consider the temporal development of waves under the boundary condition at $x = 0$,

$$\eta(0,t) = \mathrm{e}^{\mathrm{i}\Omega t} \tag{3}$$

By applying the Laplace transform to (2), the formal solution for $t \geq 0$ is written as

$$\eta(x,t) = \frac{1}{2\pi \mathrm{i}} \int_L \mathrm{e}^{h(s)x} \frac{\mathrm{d}s}{s - i\Omega} \tag{4}$$

where

$$h(s) = \frac{s}{m} + \mathrm{i}s^2(1 - 2\mathrm{i}u_0 s - 5u_0^2 s^2) \text{ and } m = x/t \tag{5}$$

The line L is a vertical line lying to the right of all singularities and the integration is from $-\mathrm{i}\infty$ to $\mathrm{i}\infty$. By carrying out the integration we could obtain the final solution. The result is shown in figure 1.

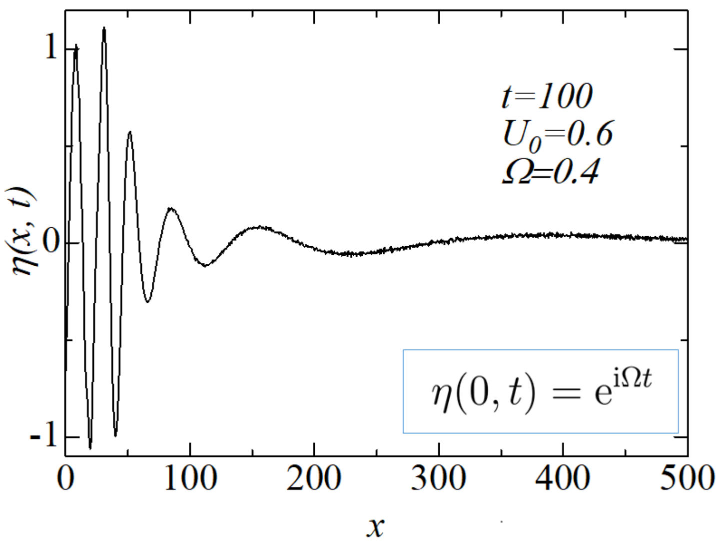

Figure 1. The development of gravity wave under the boundary condition, $\eta(0,t) = \mathrm{e}^{\mathrm{i}\Omega t}$

Next, we consider the development of capillary waves generated by small insects. By carrying out the procedure in a way similar to that for the gravity wave, we can get the characteristics of waves. The equation governing the motion can be obtained by rewriting the second term on the left-hand side of eq. (2) as $1/\sqrt{2}\left(I^{-1/2} - K^{-1/2}\right)\partial\eta/\partial t$. As a simpler case, we solve an initial- value problem to the equation. As an initial value at $x = 0$, we choose $\eta(x,0) = \sqrt{a/\pi}\,\mathrm{e}^{-ax^2}$. The development at $t = 75$ is shown in figure 2 for $a = 1$. This wave propagates far upstream containing high energy, i.e., with higher frequency but low amplitude. The charateristic is contrary to the case of gravity waves in which the wave energy is restricted near the origin of the wave disturbance.

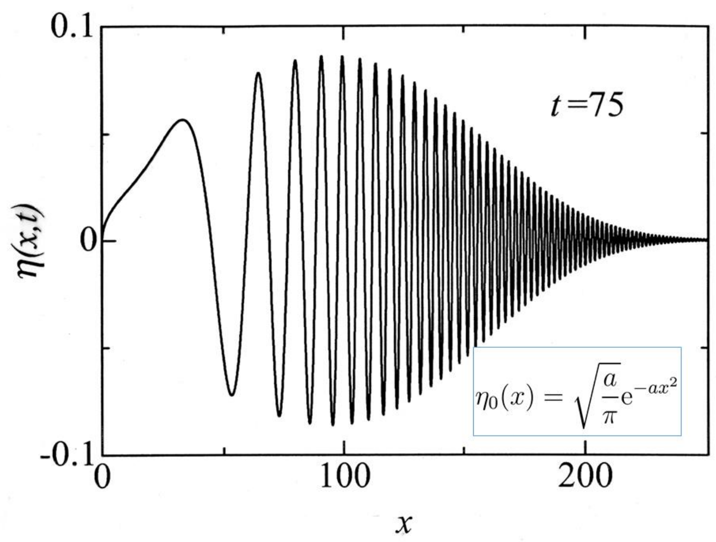

Figure 2. Asymptotic wave form for capillary wave at $t = 75$

It is known that whirligig beetles become active at night. For this reason their eyes are not useful in their night action. Our result suggests that they have a strategy that they use the wave as a signal for detecting their food. In constrast with the gravity wave, capillary ones generated by small insects have remarkable properties. It should be noted that the difference between the two kinds of waves comes from that between the phase and group velocities.

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